

FOCK SPACE REPRESENTATION OF DIFFERENTIAL CALCULUS ON THE NONCOMMUTATIVE QUANTUM SPACE

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ABSTRACT

A complete Fock space representation of the covariant differential calculus on quantum space is constructed. The consistency criteria for the ensuing algebraic structure, mapping to the canonical fermions and bosons and the consequences of the new algebra for the statistics of quanta are analyzed and discussed. The concept of statistical transmutation between bosons and fermions is introduced.

PACS Nos : 03.70; 05.30; 02.20; 71.27

1. Introduction

Quantum groups, quantum vector spaces and the underlying notion of deformations have substantially enriched the arena of mathematics and mathematical physics. Formulations of the covariant differential calculi on non-commuting ‘quantum’ spaces, and their Fock space realizations have recently attracted much attention^{1–5}. The reinterpretation of these differential calculi in terms of creation and annihilation operators have led to various generalizations of Heisenberg canonical commutation relations and enabled one to introduce particles which obey generalized quantum statistics^{6–8}.

In the present communication, it is shown that the existing scheme of mapping the differential calculus to Fock space is incomplete. Whereas the co-ordinates of the quantum plane are identified with creation operators, no such identification has been made with regard to the co-ordinates of the exterior quantum plane. We complete this scheme by introducing an additional set of creation and annihilation operators, and obtain the associated commutation relations.

The Fock space corresponding to the non-commuting differential calculus describes the states of two distinct kinds of quanta, bosonic and fermionic. A non-trivial consequence of the present formalism concerns the possibility of statistical transmutation between these bosons and fermions. This transmutation vanishes when the deformation is removed.

In the next section, we briefly recapitulate the essentials of the differential calculus on the quantum space. The various steps leading to the construction of the associated Fock space are outlined in Sec.3. This is followed by Sec.4 which is on statistical transmutation. Consistency conditions for the Fock space are discussed in Sec.5. The algebraic structure is completed in Sec.6 where we also give the relationship between the new algebra and the canonical algebra of fermions and bosons. Sec.7 is devoted to some concluding remarks.

2. Differential calculus on the quantum plane

The “quantum” space or plane is characterized by noncommuting coordinates $x_i (i = 1 \dots n)$ satisfying the q-commutation relation :

$$x_i x_j - q x_j x_i = 0, \quad \text{for } i < j \quad (1)$$

where the deformation parameter q is a real number. The differential calculus in the quantum space is constructed using three sets of basic entities, viz. (i) coordinates x_i , (ii) derivatives $\partial/\partial x_i$ and (iii) differentials or coordinates of the exterior quantum plane dx_i , together with the q -commutation relations among these. In addition to (1), the required relations are taken to be^{1,3} the following set (throughout the paper, we shall take $i < j$, in any relation involving i and j , unless otherwise specified):

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{1}{q} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} = 0 \quad (2)$$

$$\frac{\partial}{\partial x_i} x_i - q^2 x_i \frac{\partial}{\partial x_i} = 1 + (q^2 - 1) \sum_{k=i+1}^n x_k \frac{\partial}{\partial x_k} \quad (3)$$

$$\frac{\partial}{\partial x_i} x_j - q x_j \frac{\partial}{\partial x_i} = 0 \quad (4)$$

$$\frac{\partial}{\partial x_j} x_i - q x_i \frac{\partial}{\partial x_j} = 0 \quad (5)$$

$$dx_i dx_j + \frac{1}{q} dx_j dx_i = 0 \quad (6)$$

$$(dx_i)^2 = 0 \quad (7)$$

$$x_i dx_i - q^2 dx_i x_i = 0 \quad (8)$$

$$x_i dx_j - q dx_j x_i - (q^2 - 1) dx_i x_j = 0 \quad (9)$$

$$x_j dx_i - q dx_i x_j = 0 \quad (10)$$

$$\frac{\partial}{\partial x_i} dx_i - \frac{1}{q^2} dx_i \frac{\partial}{\partial x_i} - \left(\frac{1}{q^2} - 1 \right) \sum_{k=1}^{i-1} dx_k \frac{\partial}{\partial x_k} = 0 \quad (11)$$

$$\frac{\partial}{\partial x_i} dx_j - \frac{1}{q} dx_j \frac{\partial}{\partial x_i} = 0 \quad (12)$$

$$\frac{\partial}{\partial x_j} dx_i - \frac{1}{q} dx_i \frac{\partial}{\partial x_j} = 0 \quad (13)$$

We can also introduce the exterior differential d

$$d = \sum_i dx_i \frac{\partial}{\partial x_i} \quad (14)$$

It can be verified that, as a consequence of the relations (1-13), the exterior differential while operating on functions f and g of the coordinates x_i satisfies Leibnitz rule :

$$d(fg) = (df)g + f(dg) \quad (15)$$

3. Towards Fock space realization

A partial Fock space realization of the differential calculus has been constructed through the mapping^{6,8,9}

$$x_i \rightarrow b_i^\dagger \quad (16)$$

$$\frac{\partial}{\partial x_i} \rightarrow b_i \quad (17)$$

where b_i and b_i^\dagger are annihilation and creation operators and one assumes the existence of a vacuum state $|0\rangle$ annihilated by all b_i 's:

$$b_i |0\rangle = 0 \quad (18)$$

These operators satisfy the following algebra obtained from (1-5):

$$b_i^\dagger b_j^\dagger - q b_j^\dagger b_i^\dagger = 0 \quad (1')$$

$$b_i b_j - \frac{1}{q} b_j b_i = 0 \quad (2')$$

$$b_i b_i^\dagger - q^2 b_i^\dagger b_i = 1 + (q^2 - 1) \sum_{k=i+1}^n b_k^\dagger b_k \quad (3')$$

$$b_i b_j^\dagger - q b_j^\dagger b_i = 0 \quad (4')$$

$$b_j b_i^\dagger - q b_i^\dagger b_j = 0 \quad (5')$$

Note that (2') and (5') are the hermitian conjugates of (1') and (4') respectively. In order to have the complete Fock space realization of the differential calculus, one has to take the following three steps.

(A) Mapping of dx_i to a creation operator

$$dx_i \longrightarrow f_i^\dagger \quad (19)$$

Consequently, (6-13) lead to

$$f_i^\dagger f_j^\dagger + \frac{1}{q} f_j^\dagger f_i^\dagger = 0 \quad (6')$$

$$f_i^\dagger f_i^\dagger = 0 \quad (7')$$

$$b_i^\dagger f_i^\dagger - q^2 f_i^\dagger b_i^\dagger = 0 \quad (8')$$

$$b_i^\dagger f_j^\dagger - q f_j^\dagger b_i^\dagger - (q^2 - 1) f_i^\dagger b_j^\dagger = 0 \quad (9')$$

$$b_j^\dagger f_i^\dagger - q f_i^\dagger b_j^\dagger = 0 \quad (10')$$

$$b_i f_i^\dagger - \frac{1}{q^2} f_i^\dagger b_i - \left(\frac{1}{q^2} - 1\right) \sum_{k=1}^{i-1} f_k^\dagger b_k = 0 \quad (11')$$

$$b_i f_j^\dagger - \frac{1}{q} f_j^\dagger b_i = 0 \quad (12')$$

$$b_j f_i^\dagger - \frac{1}{q} f_i^\dagger b_j = 0 \quad (13')$$

Thus, using the mapping relations (16,17,19), the entire algebra of $\left\{x, \frac{\partial}{\partial x}, dx\right\}$ given by (1 - 13) has been converted to the algebra of $\{b^\dagger, b, f^\dagger\}$ given by (1' - 13').

(B) Once the creation operator f_i^\dagger has been introduced, the existence of the annihilation operator follows through the hermitian conjugation, i.e. $f = (f^\dagger)^\dagger$. As an immediate consequence, we have the following additional relations obtained by taking hermitian conjugate of (6' - 13') :

$$f_j f_i + \frac{1}{q} f_i f_j = 0 \quad (20)$$

$$f_i f_i = 0 \quad (21)$$

$$f_i b_i - q^2 b_i f_i = 0 \quad (22)$$

$$f_j b_i - q b_i f_j - (q^2 - 1) b_j f_i = 0 \quad (23)$$

$$f_i b_j - q b_j f_i = 0 \quad (24)$$

$$f_i b_i^\dagger - \frac{1}{q^2} b_i^\dagger f_i - \left(\frac{1}{q^2} - 1\right) \sum_{k=1}^{i-1} b_k^\dagger f_k = 0 \quad (25)$$

$$f_j b_i^\dagger - \frac{1}{q} b_i^\dagger f_j = 0 \quad (26)$$

$$f_i b_j^\dagger - \frac{1}{q} b_j^\dagger f_i = 0 \quad (27)$$

We assume that the vacuum state is annihilated by all the f_i 's also :

$$f_i |0\rangle = 0 \quad (28)$$

(C) In the above eqs.(1' - 13', 20 - 27), the commutation properties between all pairs of operators except the pair f and f^\dagger have been specified. In order to complete the algebraic structure for the Fock space realization, we have to know the commutation between f and f^\dagger . We shall achieve this final step in Sec.6.

4. Statistical transmutation

In this section, we examine the nature of the Fock space generated by the creation and annihilation operators satisfying the algebra given in the last section. The Fock space consists of the vacuum state $|0\rangle$ defined in (18) and (28) together with the set of states obtained by letting any product of an arbitrary number of creation operators b_i^\dagger, f_j^\dagger act on $|0\rangle$.

Because of (7'), we see that the f-quanta obey Pauli's exclusion principle while there is no such restriction on the b-quanta. Hence we shall call the f and b as fermions and bosons respectively, although they are not to be identified with the canonical fermions and bosons.

The algebra of the operators b, b^\dagger, f and f^\dagger given in Sec.3 is not invariant under the phase transformation:

$$\left. \begin{aligned} b_i &\longrightarrow e^{i\phi_i} b_i \quad ; \quad b_i^\dagger \longrightarrow e^{-i\phi_i} b_i^\dagger \\ f_i &\longrightarrow e^{i\psi_i} f_i \quad ; \quad f_i^\dagger \longrightarrow e^{-i\psi_i} f_i^\dagger \end{aligned} \right\} \quad (29)$$

where ϕ_i and ψ_i are arbitrary real numbers. As a consequence, the Fock states constructed with these operators do not have definite values for n_i and m_i , where n_i and m_i are the number of b_i and f_i quanta respectively. However, the algebra is invariant under the above transformation (29) if $\phi_i = \psi_i$. Hence the Fock states do have definite values for t_i , the total number of quanta of index i (including bosons and fermions) : $t_i = n_i + m_i$. Further, invariance of the algebra is again restored if all ϕ_i and ψ_i are independent of i and this implies that the Fock states have definite values for the total number of bosons $n = \sum_i n_i$ and the total number of fermions $m = \sum_i m_i$. However, note that if we specify t_i for all i and n, m will be redundant, since $m = \sum_i t_i - n$.

The new feature of the Fock space which allows the bosons and fermions to be transmuted into each other so that only the total t_i can be specified rather than n_i and m_i separately, can be called *statistical transmutation*. Actually, every such transmutation of a boson into fermion for a particular index always goes with the simultaneous transmutation of a fermion into boson for some other index so that the total number of bosons and the total number of fermions is conserved. Hence, considering the bosons alone, one can recognize an index transmutation also and similarly for the fermions.

Because of statistical transmutation, a multiparticle state containing bosons and fermions has a new kind of exchange property, which can be read off from (9'). For the state vector containing $b_i f_j (i < j)$, exchange of the boson b_i and the fermion f_j leads to a state that is a linear superposition of the state vector containing $f_j b_i$ with the state vector containing $f_i b_j$.

Statistical transmutation is a nontrivial complication in the construction of the new Fock space and so one may even question the existence of such a Fock space. In the next two sections we shall give an affirmative answer to this question.

5. Consistency conditions

Let c denote any of the annihilation operators b_i or f_i and c^\dagger denote b_i^\dagger or f_i^\dagger . We can classify the algebraic relations of Sec.3, into two categories (i) cc^\dagger relations (3' - 5', 11' - 13', 25-27), (ii) cc relations (2', 20-24). The $c^\dagger c^\dagger$ relations (1', 6' - 10') are just hermitian conjugates of the cc relations and hence are not independent. It is an important fact which does not seem to be well recognized, that within the framework of Fock space, the cc relations themselves are not independent of the cc^\dagger relations. For, given the vacuum state defined by eqs.(18) and (28), and the rules for the commutation of c and c^\dagger given by the cc^\dagger relations, *all* matrix elements in the Fock space can be computed. Hence any cc relation which is imposed will be either inconsistent with the cc^\dagger relations, or derivable from the cc^\dagger relations, if consistent.

To derive the cc relations from the cc^\dagger relations, we proceed as follows¹⁰. Let Q_{ij}^α be a quadratic in c 's such as the left-hand side of any of the cc relations (2', 20-24), with α denoting the equation number. By using the cc^\dagger algebra (3' - 5', 11' - 13', 25-27), we shall show that

$$Q_{ij}^\alpha c_k^\dagger = \sum_{\beta \ell m t} F_{ijk}^{\alpha \beta, \ell m t} c_t^\dagger Q_{\ell m}^\beta \quad (30)$$

where $F_{ijk}^{\alpha \beta, \ell m t}$ is some q -dependent number. Applying this equation again, we get

$$Q_{ij}^\alpha c_k^\dagger c_p^\dagger = \sum_{\beta \ell m t} \sum_{\gamma u v s} F_{ijk}^{\alpha \beta, \ell m t} F_{\ell m p}^{\beta \gamma, u v s} c_t^\dagger c_s^\dagger Q_{uv}^\gamma. \quad (31)$$

Thus, Q_{ij}^α can be pushed to the right of any string of creation operators $c_k^\dagger c_p^\dagger \dots$. Also note that the string of creation operators can contain b^\dagger and f^\dagger in arbitrary order. Allowing both sides of equations such as (30) or (31) to act on $|0\rangle$, the right-hand-side vanishes

because of (18) and (28) and so we see that Q_{ij}^α acting on any Fock state $c_k^\dagger c_p^\dagger \dots |0\rangle$ gives zero. Hence we may write the operator identity :

$$Q_{ij}^\alpha = 0 \quad (32)$$

which are the *cc* relations. Thus, (30) are the necessary and sufficient conditions for the existence of the *cc* relations and they are also the consistency conditions for the Fock space realization.

After a straightforward computation, we get the following results where $i < j$.

$$\begin{aligned} Q_{ij}^{2'} b_k^\dagger &= q^2 b_k^\dagger Q_{ij}^{2'}, \quad \text{for } k \neq i \text{ or } j \\ &= q^3 b_j^\dagger Q_{ij}^{2'} + q(q^2 - 1) \sum_{a=j+1}^n b_a^\dagger Q_{ia}^{2'}, \quad \text{for } k = j \\ &= q^3 b_i^\dagger Q_{ij}^{2'} + q(q^2 - 1) \sum_{a=i+1}^{j-1} b_a^\dagger Q_{aj}^{2'} \\ &\quad - (q^2 - 1) \sum_{a=j+1}^n b_a^\dagger Q_{ja}^{2'}, \quad \text{for } k = i \end{aligned} \quad (33)$$

$$\begin{aligned} Q_{ij}^{2'} f_k^\dagger &= \frac{1}{q^2} f_k^\dagger Q_{ij}^{2'}, \quad \text{for } k \neq i \text{ or } j \\ &= \frac{1}{q^3} f_j^\dagger Q_{ij}^{2'} - \frac{1}{q^2} (1 - q^2) \sum_{a=1}^{i-1} f_a^\dagger Q_{ai}^{2'} \\ &\quad + \frac{1}{q^3} (1 - q^2) \sum_{a=i+1}^{j-1} f_a^\dagger Q_{ia}^{2'}, \quad \text{for } k = j \\ &= \frac{1}{q^3} f_i^\dagger Q_{ij}^{2'} + \frac{1}{q^3} (1 - q^2) \sum_{a=1}^{i-1} f_a^\dagger Q_{aj}^{2'}, \quad \text{for } k = i \end{aligned} \quad (34)$$

$$\begin{aligned} Q_{ij}^{20} b_k^\dagger &= \frac{1}{q^2} b_k^\dagger Q_{ij}^{20}, \quad \text{for } k \neq i \text{ or } j \\ &= \frac{1}{q^3} b_j^\dagger Q_{ij}^{20} + \frac{1}{q^5} (1 - q^4) b_i^\dagger Q_{ii}^{21} + \frac{1}{q^3} (1 - q^2) \sum_{a=i+1}^{j-1} b_a^\dagger Q_{ia}^{20} \\ &\quad + \frac{1}{q^4} (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{ai}^{20}, \quad \text{for } k = j \\ &= \frac{1}{q^3} b_i^\dagger Q_{ij}^{20} + \frac{1}{q^3} (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{aj}^{20}, \quad \text{for } k = i \end{aligned} \quad (35)$$

$$\begin{aligned} Q_{ii}^{21} b_k^\dagger &= \frac{1}{q^2} b_k^\dagger Q_{ii}^{21}, \quad \text{for } k \neq i \\ &= \frac{1}{q^4} b_i^\dagger Q_{ii}^{21} + \frac{1}{q^3} (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{ai}^{20}, \quad \text{for } k = i \end{aligned} \quad (36)$$

$$\begin{aligned} Q_{ii}^{22} b_k^\dagger &= b_k^\dagger Q_{ii}^{22}, \quad \text{for } k \neq i \\ &= b_i^\dagger Q_{ii}^{22} + (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{ai}^{24} + \frac{1}{q} (q^2 - 1) \sum_{a=i+1}^n b_a^\dagger Q_{ia}^{24}, \quad \text{for } k = i \end{aligned} \quad (37)$$

$$\begin{aligned}
Q_{ij}^{23} b_k^\dagger &= b_k^\dagger Q_{ij}^{23}, \quad \text{for } k \neq i \text{ or } j \\
&= \frac{1}{q} b_j^\dagger Q_{ij}^{23} + \frac{1}{q} (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{ai}^{24} + \frac{1}{q} (1 - q^2) b_i^\dagger Q_{ii}^{22} \\
&\quad + \frac{1}{q} (1 - q^2) \sum_{a=i+1}^{k-1} b_a^\dagger Q_{ia}^{23}, \quad \text{for } k = j \\
&= q b_i^\dagger Q_{ij}^{23} - \frac{1}{q^2} (q^2 - 1)^2 \sum_{a=1}^{i-1} b_a^\dagger Q_{aj}^{24} + \frac{1}{q^2} (q^2 - 1) b_j^\dagger Q_{ii}^{22} \\
&\quad + \frac{1}{q} (q^2 - 1) \sum_{a=i+1}^{j-1} b_a^\dagger Q_{ja}^{23} + \frac{1}{q} (q^2 - 1) \sum_{a=j+1}^n b_a^\dagger Q_{ja}^{24}, \quad \text{for } k = i
\end{aligned} \tag{38}$$

$$\begin{aligned}
Q_{ij}^{24} b_k^\dagger &= b_k^\dagger Q_{ij}^{24}, \quad \text{for } k \neq i \text{ or } j \\
&= q b_j^\dagger Q_{ij}^{24} + \frac{1}{q} (q^2 - 1) \sum_{a=j+1}^n b_a^\dagger Q_{ia}^{24}, \quad \text{for } k = j \\
&= \frac{1}{q} b_i^\dagger Q_{ij}^{24} + \frac{1}{q} (1 - q^2) \sum_{a=1}^{i-1} b_a^\dagger Q_{aj}^{24}, \quad \text{for } k = i
\end{aligned} \tag{39}$$

We see that all the above equations are of the form (30). This is not the complete set of consistency conditions; to get these we will need the commutation properties between f and f^\dagger which are yet to be obtained. However, the validity of the consistency conditions (33-39) already points to the existence of an underlying Fock space and encourages us to find it.

6. Completion of the Fock space realization

We find the required Fock space by showing that the b, f system is in fact related to the canonical bose-fermi system \tilde{b}, \tilde{f} defined by the usual algebra (for all i and k):

$$[\tilde{b}_i, \tilde{b}_k^\dagger] = \delta_{ik} \quad ; \quad [\tilde{b}_i, \tilde{b}_k] = 0 \tag{40}$$

$$\{\tilde{f}_i, \tilde{f}_k^\dagger\} = \delta_{ik} \quad ; \quad \{\tilde{f}_i, \tilde{f}_k\} = 0 \tag{41}$$

$$[\tilde{b}_i, \tilde{f}_k^\dagger] = 0 \quad ; \quad [\tilde{b}_i, \tilde{f}_k] = 0 \tag{42}$$

and their hermitian conjugates where $[x, y]$ and $\{x, y\}$ denote the usual commutator and anticommutator respectively. The relationship is given by the transformation equations :

$$b_i^\dagger = q^{\sum_{k>i} \tilde{N}_k} \left(\frac{[\tilde{N}_i]}{\tilde{N}_i} \right)^{1/2} \tilde{b}_i^\dagger \tag{43}$$

$$f_i^\dagger = q^{\sum_{p<i} \tilde{M}_p - \sum_p \tilde{N}_p - \tilde{N}_i} \tilde{f}_i^\dagger + (1 - q^2) \sum_{k<i} q^{\sum_{p<k} \tilde{M}_p - \sum_p \tilde{N}_p - \sum_{k \leq p \leq i} \tilde{N}_p} \left(\frac{[\tilde{N}_k + 1][\tilde{N}_i]}{(\tilde{N}_k + 1)(\tilde{N}_i)} \right)^{1/2} \tilde{b}_i^\dagger \tilde{b}_k \tilde{f}_k^\dagger \tag{44}$$

and their hermitian conjugates. The operators \tilde{N}_i and \tilde{M}_i are the number operators of the \tilde{b}, \tilde{f} system :

$$\tilde{N}_i = \tilde{b}_i^\dagger \tilde{b}_i \quad ; \quad \tilde{M}_i = \tilde{f}_i^\dagger \tilde{f}_i \tag{45}$$

and they satisfy the usual commutation relations:

$$[\tilde{N}_i, \tilde{N}_k] = [\tilde{M}_i, \tilde{M}_k] = [\tilde{N}_i, \tilde{M}_k] = 0. \quad (46)$$

$$[\tilde{N}_i, \tilde{b}_k^\dagger] = \delta_{ik} \tilde{b}_k^\dagger \quad ; \quad [\tilde{M}_i, \tilde{b}_k^\dagger] = 0 \quad (47)$$

$$[\tilde{M}_i, \tilde{f}_k^\dagger] = \delta_{ik} \tilde{f}_k^\dagger \quad ; \quad [\tilde{N}_i, \tilde{f}_k^\dagger] = 0 \quad (48)$$

In (43) and (44) the square bracket enclosing a single object [L] is defined by

$$[L] = \frac{q^{2L} - 1}{q^2 - 1} = 1 + q^2 + q^4 + \dots q^{2(L-1)} \quad (49)$$

Using (43) and (44), the algebra given by (1'-13', 20-27) can be verified by straightforward but long computations.

The transformations (43) and (44) can be inverted (for $q \neq 0$) to give

$$\tilde{b}_i^\dagger = q^{-\sum_{k>i} \tilde{N}_k} \left(\frac{\tilde{N}_i}{[\tilde{N}_i]} \right)^{1/2} b_i^\dagger \quad (50)$$

$$\tilde{f}_i^\dagger = q^{-\sum_{p<i} \tilde{M}_p + \sum_p \tilde{N}_p + \tilde{N}_i} f_i^\dagger + q^{2i}(q^2 - 1) q^{-\sum_{p<i} \tilde{M}_p + \sum_{p<i} \tilde{N}_p - \sum_{p>i} \tilde{N}_p} \sum_{k<i} q^{-2k} b_i^\dagger b_k f_k^\dagger \quad (51)$$

The transformation given by (43) and (44) is our central result. This establishes the complete Fock space of the b, f system, since the latter has been expressed in terms of the \tilde{b}, \tilde{f} system which operates on the canonical Fock space of bosons and fermions.

Nevertheless, one might still want to know the q -commutation relations between f and f^\dagger . These are now derivable from (43) and (44). After a long computation we get

$$f_i f_j^\dagger + q f_j^\dagger f_i = q(1 - q^2) b_j^\dagger b_i A_{ij} \quad \text{for } i < j \quad (52)$$

$$f_i f_i^\dagger + f_i^\dagger f_i = B_i - (1 - q^2)^2 \sum_{\substack{k<i, k'<i \\ k \neq k'}} \tilde{f}_k^\dagger \tilde{f}_{k'}^\dagger \tilde{b}_k^\dagger \tilde{b}_{k'} C_{ik k'} \quad (53)$$

where A, B and C are functions of the number operators \tilde{N}_i and \tilde{M}_i :

$$A_{ij} = \left(q^{-2\sum_{k\geq i} \tilde{N}_k} + (1 - q^2) \sum_{k<i} [\tilde{N}_k + \tilde{M}_k] q^{-2\sum_{p\geq k} \tilde{N}_p} \right) q^{2\sum_{p<i} \tilde{M}_p - 2\sum_p \tilde{N}_p} \quad (54)$$

$$\begin{aligned} B_i &= q^{2\sum_{p<i} \tilde{M}_p - 2\sum_p \tilde{N}_p - 2\tilde{N}_i} \\ &+ \sum_{k<i} (1 - q^2)^2 \left\{ [\tilde{N}_k][\tilde{N}_i + 1] - (1 - q^2)^{-1} \tilde{M}_k (q^{2\tilde{N}_i} - q^{2\tilde{N}_k}) \right\} q^{2\sum_{p<k} \tilde{M}_p - 2\sum_{k\leq p\leq i} \tilde{N}_p - 2\sum_p \tilde{N}_p} \end{aligned} \quad (55)$$

$$C_{ikk'} = \left(\frac{[\tilde{N}_k][\tilde{N}_{k'} + 1]}{\tilde{N}_k(\tilde{N}_{k'} + 1)} \right)^{1/2} q^{2\tilde{N}_i + (\sum_{p < k'} + \sum_{p < k})\tilde{M}_p - (\sum_p + \sum_{k \leq p \leq i} + \sum_{k' \leq p \leq i})\tilde{N}_p} \quad (56)$$

The right hand side of (53) contains the canonical operators \tilde{b}, \tilde{f} and these can be reexpressed in terms of b, f using the inverse relations (50) and (51); we have not written them in that form since the expressions would be longer.

Further, one may like to specify the commutation relations of \tilde{N}_i and \tilde{M}_i with respect to b, f, b^\dagger and f^\dagger since these will be required for the closure of the algebra. These are (for all i and k)

$$[\tilde{N}_i, b_k^\dagger] = \delta_{ik} b_k^\dagger \quad (57)$$

$$[\tilde{M}_i, b_k^\dagger] = 0 \quad (58)$$

$$[\tilde{M}_i, f_k^\dagger] = \delta_{ik} f_k^\dagger - (1 - q^2) \sum_m D_{imk} b_k^\dagger b_m f_m^\dagger \quad (59)$$

$$[\tilde{N}_i, f_k^\dagger] = (1 - q^2) \sum_m D_{imk} b_k^\dagger b_m f_m^\dagger \quad (60)$$

together with the hermitian conjugates of these relations, where

$$D_{ikm} = \delta_{ik} \theta_{mk} q^{2(k-m)-2 \sum_{p \geq k} \tilde{N}_p} - \theta_{ik} \left\{ \delta_{mi} q^{2-2 \sum_{p > i} \tilde{N}_p} - \theta_{mi} (1 - q^{2(\tilde{N}_i + 1)}) q^{2(i-m)-2 \sum_{p \geq i} \tilde{N}_p} \right\} \quad (61)$$

In eq.(61), θ_{ik} is defined to be 1 for $i < k$ and zero otherwise.

Although we have given the ff^\dagger relations (52,53) supplemented by (57-61) for the sake of exhibiting the complete algebra of b, f system and thus completing the Fock space representation of the differential calculus on the noncommuting quantum space, the alternative way of expressing our result in terms of the transformation equations (43) and (44) is simpler and more transparent.

In particular, the origin of the transmutations discussed in Sec.4 becomes clear now. Eq.(44) shows that the fermionic operator f_i^\dagger creates not only fermion \tilde{f}_i but also the boson \tilde{b}_i , at the same time converting the boson \tilde{b}_k into fermion \tilde{f}_k for all $k < i$.

Relations of the type (43) which can be called generalized Klein-Jordan-Wigner relations are known from the earlier literature^{11,12} but the relation (44) which leads to the idea of transmutation is new.

7. Discussion

We have constructed here the complete Fock space associated with the differential calculus on the quantum space. The algebraic relations between the creation and annihilation operators spanning the Fock space have been derived and their internal consistency established.

The present formalism leads to the notion of statistical transmutation between different kinds of quanta residing in the generalized Fock space. Consequently, the number operators

for individual quanta are not conserved, only certain partial sums of number operators are conserved.

We have been able to map the entire set of new creation and annihilation operators to the creation and annihilation operators for canonical fermions and bosons. Because of the existence of such transformations, we can say that as far as the underlying Fock space is concerned, the deformations leading to covariant differential calculus do not lead to anything fundamentally new. What one gets is only a different avatar of the canonical algebra of fermions and bosons¹⁰, however with statistical transmutation. Thus the formalism presented here demystifies the non-commutative differential calculus on which so much recent work has been done, by providing its representation in a linear vector space which is a composite of canonical fermionic and bosonic spaces.

The insight gained through this understanding of the noncommutative differential calculus in terms of Fock space may prove useful for further lines of investigation. Some of these are :

(a) Is it possible to have a nontrivial deformation of the differential calculus that does not require the statistical transmutation ? This may require the dropping of the Leibnitz rule (15).

(b) More general transmutations can be introduced if we replace (43) and (44) by

$$b_i^\dagger = F_i \tilde{b}_i^\dagger + \sum_j G_{ij} \tilde{f}_i^\dagger \tilde{f}_j \tilde{b}_j^\dagger \quad (62)$$

$$f_i^\dagger = H_i \tilde{f}_i^\dagger + \sum_j K_{ij} \tilde{b}_i^\dagger \tilde{b}_j \tilde{f}_j^\dagger \quad (63)$$

where \tilde{b}_i and \tilde{f}_i are canonical boson and fermion annihilation operators and F_i, G_{ij}, H_i and K_{ij} are functions of the canonical number operators as well as one or more deformation parameters. Hence, the path is open, to construct a variety of new deformed differential calculi. One may also remark that transformations of the type (62) and (63) can be used to describe the transmutation of any two species of canonical quanta ; both may be bosons or fermions.

Differential calculus is generated by three operators $(x, \frac{\partial}{\partial x}, dx)$, whereas Fock space is spanned by $(b^\dagger, b, f^\dagger, f)$. What is the significance of the additional operator f , with regard to differential calculus? A possible answer may be provided in the context of the Lagrangian and Hamiltonian dynamics in quantum space¹⁴ in which the q -deformed differential calculus plays an essential role. One may introduce the velocity \dot{x} as the differential dx divided by dt where t , the time, is taken to be a commuting number. The dynamical formalism will require the calculus to be extended to the derivative $\frac{\partial}{\partial \dot{x}}$. For this extended calculus, one may have the mapping :

$$x \rightarrow b^\dagger \quad ; \quad \frac{\partial}{\partial x} \rightarrow b \quad ; \quad \dot{x} \rightarrow f^\dagger \quad ; \quad \frac{\partial}{\partial \dot{x}} \rightarrow f. \quad (64)$$

Finally we note that the transformations linking the new operators to canonical operators become ill-defined when $q \rightarrow 0$ or $\pm\infty$. For such singular values of q , new kinds of statistics ('null statistics') living in new Fock spaces ('Fock spaces of frozen order') emerge¹³. Statistical transmutation can be incorporated into the null statistics too, leading to newer structures.

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